IOPscience iopscience.iop.org

Home Search Collections Journals About Contact us My IOPscience

Characterization of a periodically driven chaotic dynamical system

This article has been downloaded from IOPscience. Please scroll down to see the full text article.

1997 J. Phys. A: Math. Gen. 30 371

(http://iopscience.iop.org/0305-4470/30/2/005)

View the table of contents for this issue, or go to the journal homepage for more

Download details:
IP Address: 171.66.16.110
The article was downloaded on 02/06/2010 at 06:02

Please note that terms and conditions apply.

J. Phys. A: Math. Gen. **30** (1997) 371–383. Printed in the UK PII: S0305-4470(97)74909-6

Characterization of a periodically driven chaotic dynamical system

A Crisanti†, M Falcioni‡, G Lacorata§, R Purini§ and A Vulpiani†

† Dipartimento di Fisica, Università 'La Sapienza', I-00185 Roma, Italy and Istituto Nazionale di Fisica della Materia, Sezione di Roma
‡ Dipartimento di Fisica, Università 'La Sapienza', I-00185 Roma, Italy and Istituto Nazionale di Fisica Nucleare, Sezione di Roma
§ Istituto di Fisica dell' Atmosfera - CNR, P.le L. Sturzo 31 I-00144 Roma, Italy

Received 23 May 1996, in final form 10 September 1996

Abstract. We discuss how to characterize the behaviour of a chaotic dynamical system depending on a parameter that varies periodically in time, such that the time scale of the periodic variation of the parameter is much larger than the 'internal' time scale. In particular, we study the predictability time, the correlations and the mean responses, by defining a local-in-time version of these quantities. We find that the local quantities strongly depend on the phase of the cycle. In this case, the standard global quantities can give misleading information.

1. Introduction

The forecast of the behaviour of a system when its evolution law is known is a problem with an obvious interest in many fields of scientific research. Roughly speaking, within this problem two main areas of investigation may be identified.

(A) The definition of the 'predictability time'. If one knows the initial state of a system, with a precision $\delta_0 = |\delta \boldsymbol{x}(0)|$, what is the maximum time T_{p} within which one is able to know the system future state with a given tolerance $\delta_{\max}$?

(B) The understanding of the relaxation properties. What is the relation between the mean response of a system to an external perturbation and the features of its unperturbed state [1]? Using the terminology of statistical mechanics, one wants to reduce 'non-equilibrium' properties, such as relaxation and responses, to 'equilibrium' ones, such as correlation functions [2, 3].

A remarkable example of a type-A problem is weather forecasting, where one has to estimate the maximum time for which the prediction is accurate enough. As an example of a type-B problem, of geophysical interest, one can mention a volcanic eruption which induces a practically instantaneous change in the temperature. In this case it is relevant to understand how the difference between the atmospheric state after the eruption and the hypothetical unperturbed state without the eruption evolves in time. In practice, one wants to understand how a system absorbs, on average, the perturbation $\delta f(\tau)$ of a certain quantity f—for example the temperature—just by looking at the statistical features, as correlations, of f in the unperturbed regime. This is the so-called fluctuation/relaxation problem [4].

As far as problem A is concerned, in the presence of deterministic chaos, a rather common situation, the distance between two initially close trajectories diverges

 © 1997 IOP Publishing Ltd

exponentially:

$$|\delta \boldsymbol{x}(t)| \sim \delta_0 \exp(\lambda t) \tag{1}$$

where λ is the maximum Lyapunov exponent of the system [5]. Note that (1) just gives the physical meaning of the maximum Lyapunov exponent, a more precise definition is

$$\lambda = \lim_{t\to\infty} \lim_{\delta_0\to 0} \frac{1}{t} \ln \frac{|\delta \boldsymbol{x}(t)|}{\delta_0}. \tag{2}$$

From (1) it follows that

$$T_\mathrm{p} \sim \frac{1}{\lambda} \ln\left(\frac{\delta_\mathrm{max}}{\delta_0}\right). \tag{3}$$

Since the dependence on δ_max and δ_0 is very weak, T_p appears to be proportional to the inverse of the Lyapunov exponent. We stress, however, that (3) is just a naive answer to the predictability problem, since it does not take into account the following relevant features of the chaotic systems.

(i) The Lyapunov exponent is a global quantity, i.e. it measures the average exponential rate of divergence of nearby trajectories. In general, there are finite-time fluctuations of this rate, described by means of the so-called *effective* Lyapunov exponent $\gamma_t(\tau)$. This quantity depends on both the time delay τ and the time t at which the perturbation acted [6]. Therefore, the predictability time T_p fluctuates, following the γ-variations [7].

(ii) In systems with many degrees of freedom one has to understand how a perturbation grows and propagates through the different degrees of freedom [8]. For example, one can be interested in the prediction on certain variables, for example those associated with large scales in weather forecasting, while the perturbations act on a different set of variables, for example those associated to small scales. For an analysis of this problem in the case of turbulence see [9].

(iii) If one is interested in non-infinitesimal perturbations, and the system possesses many characteristic times, such as the eddy turn-over times in fully developed turbulence, then T_p is determined by the detailed mechanism of propagation of the perturbations through different degrees of freedom, due to nonlinear effects. In particular, T_p may have no relation with λ [9].

An interesting particular case where observation (i) applies is given by a system whose evolution is ruled by a set of differential equations,

$$\mathrm{d}\boldsymbol{x}/\mathrm{d}t = \boldsymbol{f}(\boldsymbol{x}, t) \tag{4}$$

which depend periodically on time,

$$\boldsymbol{f}(\boldsymbol{x}, t + T) = \boldsymbol{f}(\boldsymbol{x}, t). \tag{5}$$

In this case one can have a kind of 'seasonal' effect, for which the system shows an alternation, roughly periodic, of low and high predictability. This happens, for example, in the recently studied case of stochastic resonance in a chaotic deterministic system, where one observes a roughly periodic sequence of chaotic and regular evolution intervals [10].

As far as problem B is concerned, it is possible to show that, in a chaotic system with an invariant measure $P(\boldsymbol{x})$, there exists a relation between the mean response $\langle \delta x_j(\tau)\rangle_P$ after a time τ from a perturbation $\delta x_i(0)$, and a suitable correlation function [3]. Namely, one has the following equation,

$$R_{ij}(\tau) \equiv \frac{\langle \delta x_j(\tau)\rangle_P}{\delta x_i(0)} = \left\langle x_j(\tau) \frac{\partial S(\boldsymbol{x}(0))}{\partial x_i} \right\rangle_P \tag{6}$$

where $S(\boldsymbol{x}) = -\ln P(\boldsymbol{x})$. Equation (6) ensures that the mean relaxation of the perturbed system is equal to some correlation of the unperturbed system. Of course, since, in general, one does not know $P(\boldsymbol{x})$, equation (6) provides only a qualitative information.

In this paper we discuss how one has to reformulate the predictability problem, both of type A and of type B, for systems where the 'seasonal' effects are important, i.e. when some relevant characteristic times change, in a substantial way, periodically in time.

In section 2 we discuss how to approach the predictability problem in systems with time periodic evolution laws and such that one can identify two time scales. Section 3 is devoted to the study of a toy model whose behaviour, in spite of the simplicity of the model, catches the basic features of a system with seasonal effects. In section 4 we show the results of numerical experiments that illustrate the relevance of the concepts introduced in section 2.

2. The characterization of systems with periodic effects

We consider systems in which one can identify two time scales: the first, that we call T_E, due to the coupling with an external time-dependent driver; the second, an 'internal' time scale T_I, characterizes the system in the limit of a constant external coupling, which we call the 'stationary limit'. In the following, the external time dependence will be assumed periodic with period T, so that $T_E = T$.

If the system, in the stationary limit, is chaotic, one can take as the internal time scale the Lyapunov time, i.e. the inverse of the maximum Lyapunov exponent, $T_I \sim 1/\lambda$.

In the case $T_E \gg T_I$ one can assume that the system is adiabatically driven through different dynamical regions, so that for observation times short with respect to the long external time it evolves on a local (in time) attractor. If during its tour across the slowly changing phase space the system visits regions where the effective Lyapunov exponents are different enough, then the system shows up sensibly different predictability times that may occur regularly when the driver is time periodic. Consider, for instance, a slight modification [10] of the Lorenz model [11], which is the first geophysical dynamical system where deterministic chaos has been observed:

$$\begin{aligned} dx/dt &= 10(y - x) \\ dy/dt &= -xz + r(t)x - y \\ dz/dt &= xy - \tfrac{8}{3}z \end{aligned} \tag{7}$$

where the control parameter has a periodic time variation

$$r(t) = r_0 - A\cos(2\pi t/T). \tag{8}$$

Since this model describes the convection of a fluid heated from below between two layers whose temperature difference is proportional to the Rayleigh number r, the periodic variations of r roughly mimic the seasonal changing on the solar heat inputs. A good approximation of the solution for very large T may be given by

$$x(t) = y(t) = \pm\sqrt{\tfrac{8}{3}(r(t) - 1)} \qquad z(t) = r(t) - 1 \tag{9}$$

which is obtained from the fixed points of the standard Lorenz model by replacing r with $r(t)$. The stability of this solution is a rather complicated issue, which depends on the values of r_0, A and T. It is natural to expect that if r_0 is larger than r_{cr}—the value for which, in the standard Lorenz model, a transition takes place from stable fixed points to a chaotic attractor—the solution is unstable. In this case, for A large enough (at least $r_0 - A < r_{cr}$) one has a mechanism similar to stochastic resonance in bistable systems with random forcing [10]. The value of T is crucial. For large T the systems behave as follows. If

$$T_n \simeq nT/2 - T/4 \qquad (n = 1, 2, \ldots) \tag{10}$$

are the times at which $r(t) = r_{cr}$, one can naively expect that for $0 < t < T_1$, when $r(t)$ is smaller than r_{cr}, the system is stable and the trajectory is close to one of the two solutions (9), while for $T_1 < t < T_2$, when $r(t) > r_{cr}$, both solutions (9) are unstable and the trajectory relaxes toward a sort of 'adiabatic' chaotic attractor. The chaotic attractor smoothly changes at varying of r above the threshold r_{cr}, but if T is large enough this dependence to first approximation can be neglected. When $r(t)$ becomes again smaller than r_{cr}, the 'adiabatic' attractor disappears and, in general, the system is far from the stable solutions (9), but it relaxes toward them, being attractive. If the half-period is much larger than the relaxation time, in general, the system follows one of the two regular solutions (9) for $T_{2n+1} < t < T_{2n+2}$. However, there is a small but non-zero probability that the system has not enough time to relax to (9) and its evolution remains irregular. Figure 1 shows the time evolution of the variable z, for $r_0 = 25.5$ and $A = 4$, in the cases $T = 300$ (a) and $T = 1600$ (b). They provide unambiguous numerical evidence that the jumps from the irregular to the regular behaviour (and *vice versa*) are well synchronized with $r(t)$, with a probability close to unity when the forcing period T is very long (see figure 1(b)). On the other hand, for a small value of T the system often does not perform the transition from the irregular to the regular behaviour, as seen in figure 1(a).

It is worth stressing that the system is chaotic. In both cases, in fact, the first Lyapunov exponent is positive.

It is rather clear from this example that the Lyapunov exponent is not able to characterize the above behaviour, since it just refers to a very long time property of the system, i.e. a property involving times longer than T. A more useful and detailed information can be obtained by computing a 'local' average of the exponential rate of divergence for initially close trajectories. By this we mean an average which explicitly depends on the time t_0, modulus the external period T, to test the behaviour of the system in the different states of the external driver. In this way one can make evident different behaviours, if any, of the system.

We, therefore, define the mean effective Lyapunov exponent, for the time $t_0 \in [0, T]$ and for a delay τ, as

$$\langle \gamma(\tau) \rangle_{t_0} = \lim_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N-1} \gamma(t_0 + kT, \tau) \tag{11}$$

where

$$\gamma(t, \tau) = \frac{1}{\tau} \ln \frac{|\delta \boldsymbol{x}(t+\tau)|}{|\delta \boldsymbol{x}(t)|} \tag{12}$$

is the local expansion rate, and $\delta \boldsymbol{x}(t)$ evolves according to the linear equation

$$\frac{\mathrm{d}}{\mathrm{d}t} \delta x_i(t) = \sum_j \frac{\partial f_i(\boldsymbol{x}(t), t)}{\partial x_j} \delta x_j. \tag{13}$$

From this definition it is clear that $\langle \gamma(\tau) \rangle_{t_0}$ measures the growth of the distance after a time delay τ between two trajectories that differ by $|\delta \boldsymbol{x}(t)|$ when the external driver passes through a fixed value (or state). The maximum Lyapunov exponent of the system gives the global average of $\langle \gamma(\tau) \rangle_{t_0}$:

$$\lambda = \lim_{\tau\to\infty} \langle \gamma(\tau) \rangle_{t_0} = \frac{1}{T} \int_0^T \langle \gamma(\tau) \rangle_{t_0} \, \mathrm{d}t_0. \tag{14}$$

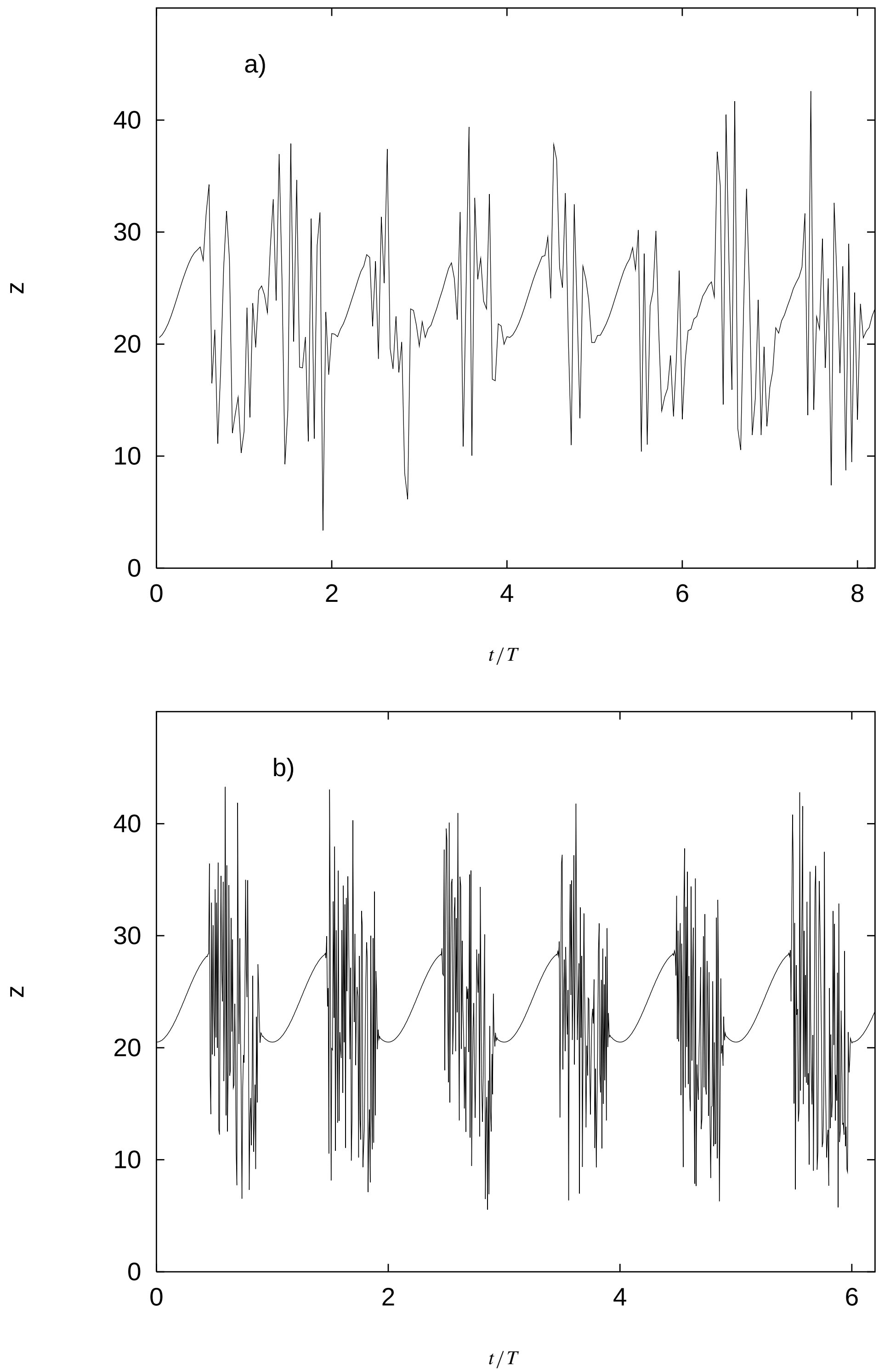

Figure 1. The variable z as a function of t/T for the model (7) and (8) with $r_0 = 25.5$, $A = 4$ and (a) $T = 300$ and (b) $T = 1600$.

If one is interested on predictability for times much smaller then T, $\langle\gamma(\tau)\rangle_{t_0}$ with $\tau \ll T$ is a more appropriate quantity than λ, since it distinguishes among different regimes. For example, in the system (8) discussed above, for the given values of the parameters, one has $\lambda > 0$, but $\langle\gamma(\tau)\rangle_{t_0} < 0$ when $t_0 \in [(n-1/4)T, (n+1/4)T]$. In the case of weather forecasting, different values of γ for different t_0 correspond to different degrees of predictability during the year.

As far as the response properties are concerned, we expect that in a chaotic system the hypothesis of the existence of 'adiabatic' attractors implies that a fluctuation/relaxation relation holds also as a time local property, provided one uses correlation and response functions computed according to a local, not a global, average. So besides the usual correlation function between the variables x_i and x_j,

$$C_{ij}^{(G)}(\tau) = \overline{x_i(t)x_j(t+\tau)} - \overline{x_i}\,\overline{x_j} \tag{15}$$

where $\overline{(\cdot)}$ indicates the global average,

$$\overline{A_i} = \lim_{t\to\infty} \frac{1}{t}\int_0^t A_i(t')\,\mathrm{d}t' \tag{16}$$

we introduce their correlation on a delay τ after the time $t_0 \in [0, T]$:

$$C_{ij}(t_0, \tau) = \langle x_i(t_0)x_j(t_0+\tau)\rangle_{t_0} - \langle x_i\rangle_{t_0}\langle x_j\rangle_{t_0} \tag{17}$$

where the local average is defined as in (11):

$$\langle A\rangle_{t_0} = \lim_{N\to\infty} \frac{1}{N}\sum_{k=0}^{N-1} A(t_0 + kT). \tag{18}$$

In a similar way, one can consider two different kinds of mean response function of the variable x_i to a small perturbation δx_j: the global average response,

$$R_{ij}^{(G)}(\tau) = \overline{\frac{\delta x_i(t+\tau)}{\delta x_j(t)}} \tag{19}$$

and the local average response for the time t_0,

$$R_{ij}(t_0, \tau) = \frac{\langle\delta x_i(t_0+\tau)\rangle_{t_0}}{\delta x_j(t_0)}. \tag{20}$$

The quantity (19) gives the mean response, after a delay τ, to a perturbation occurred at the time t, chosen at random, i.e. with uniform distribution in $[0, T]$. We shall see that $R_{ij}^{(G)}(\tau)$ can be rather different, even at a qualitative level, from $R_{ij}(t_0, \tau)$.

3. A toy model

The ideas discussed in section 2 are illustrated here by means of a simple model. Consider the Langevin equation [12]

$$\frac{\mathrm{d}}{\mathrm{d}t}q(t) = -a(t)q(t) + \xi(t) \tag{21}$$

where $\xi(t)$ is δ-correlated white noise, i.e. $\xi(t)$ is a Gaussian variable with

$$\langle\xi(t)\rangle = 0 \qquad \langle\xi(t)\xi(t')\rangle = 2\Gamma\,\delta(t-t') \tag{22}$$

and the coefficient $a(t)$ is a periodic function of period T: $a(t+T) = a(t)$. We require that

$$\int_0^T \mathrm{d}t\, a(t) > 0 \tag{23}$$

to ensure a well defined asymptotic probability distribution for the stochastic process given by equation (21). Moreover, we assume a slow variation of $a(t)$, i.e.

$$\min_t a(t) \gg \frac{1}{T} \tag{24}$$

so that, by making an adiabatic approximation, a 'local' probability distribution exists at any time.

Without the noise term, the process described by equation (21) is non-chaotic. Therefore, the model (21) cannot exhibit the full rich behaviour of chaotic systems, nevertheless it catches some of the relevant features. It is easy to see that the characteristic decay time of the local correlation

$$\begin{aligned} C(t_0,\tau) &= \langle q(t_0)q(t_0+\tau)\rangle_{t_0} \\ &= \lim_{N\to\infty} \frac{1}{N}\sum_{k=0}^{N-1} q(kT+t_0+\tau)q(kT+t_0) \end{aligned} \tag{25}$$

depends on t_0. This can be easily computed by using the formal solution of (21)

$$q(t) = G(t)\left[q(0) + \int_0^t \mathrm{d}\tau\, G^{-1}(\tau)\xi(\tau)\right] \tag{26}$$

where

$$G(t) = \exp\left[-\int_0^t \mathrm{d}\tau\, a(\tau)\right]. \tag{27}$$

A straightforward calculation leads to

$$C(t_0,\tau) = C(t_0,0)G(t_0,\tau)/G(t_0) \tag{28}$$

where the equal time correlation is

$$C(t_0,0) = G^2(t_0)\left[\lim_{N\to\infty}\frac{1}{N}\sum_{k=0}^{N-1} q(kT)^2 + 2\Gamma\int_0^{t_0} \mathrm{d}\tau\, G^{-2}(\tau)\right]. \tag{29}$$

In figure 2 we show $C(t_0,\tau)/C(t_0,0)$ as a function of τ for

$$a(t) = a + b\cos\left(\frac{2\pi}{T}t\right) \qquad a+b>0 \tag{30}$$

for two different values of t_0, namely $t_0 = 0$ and $t_0 = T/2$, with $T = 10$, $a = 1$, $b = -0.9$ and $\Gamma = 0.5$. The different behaviour is evident. By defining a characteristic time as the time s it takes to have $C(t_0, s) = 0.1$, we get for this case $s_0 \approx 3.475$ and $s_{T/2} \approx 1.275$. When starting from $t_0 = T/2$ the decay is almost a factor 3 faster than starting from $t_0 = 0$. The usual global average,

$$C^{(\mathrm{G})}(\tau) = \lim_{t\to\infty}\frac{1}{t}\int_0^t \mathrm{d}t'\, q(t')q(t'+\tau) \tag{31}$$

gives an average correlation function, so its characteristic decay time is not able to distinguish different regimes. Moreover, while $C(t_0,\tau)/C(t_0,0)$ does not depend on the noise strength Γ, $C^{(\mathrm{G})}(\tau)/C^{(\mathrm{G})}(0)$ does. In figure 2 we used $\Gamma = 0.5$.

We consider now how the system responds at time $t_0+\tau$ to a perturbation performed at time t_0. This is described by the mean response function $R(t_0,\tau)$, which can be computed as follows. One takes two trajectories differing at time t_0 by a quantity ϵ, i.e. $\delta q(kT+t_0) = \epsilon$

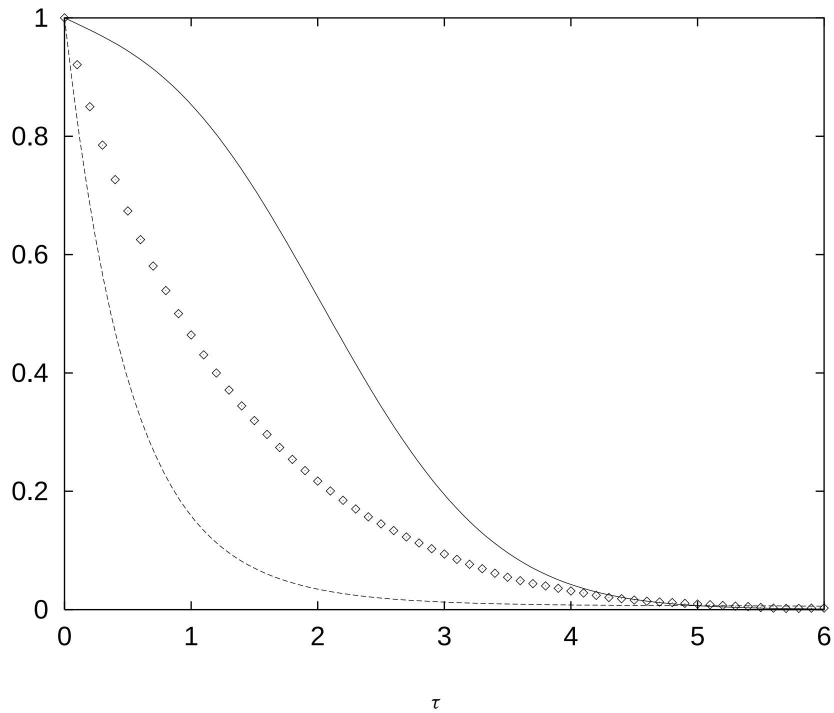

Figure 2. Model (21) and (30) with $a = 1$, $b = -0.9$, $T = 10$ and $\Gamma = 0.5$. The different symbols refer to $C^{(G)}(\tau)$ (diamonds) and $C(t_0, \tau)/C(t_0, 0)$ as functions of τ for $t_0 = 0$ (full curve) and $t_0 = T/2 = 5$ (dashed curve).

for any k, and evolving with the *same* realization of noise. Then the local response function is

$$R(t_0, \tau) = \lim_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N-1} \frac{\delta q(kT + t_0 + \tau)}{\delta q(kT + t_0)}$$
$$= \lim_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N-1} \frac{\delta q(kT + t_0 + \tau)}{\epsilon} \tag{32}$$

where $\delta q(kT + t_0 + \tau)$ is the difference between the two trajectories at time $t_0 + \tau$. Both times t_0 and τ run over a cycle, i.e. in the interval $[0, T]$.

By making use of (26) it is easy to see that

$$R(t_0, \tau) = \frac{G(t_0, \tau)}{G(t_0)}. \tag{33}$$

By combining equations (28) and (33) we have the fluctuations/relaxation relation [2]

$$C(t_0, \tau) = C(t_0, 0)\, R(t_0, \tau). \tag{34}$$

The scenario just described remains basically valid for the nonlinear Langevin equation. Consider, for example, the equation

$$\frac{\mathrm{d}}{\mathrm{d}t} q(t) = -a(t) q^3(t) + \xi(t) \tag{35}$$

where $a(t)$ is still given by (30).

It is natural to expect that, because of the adiabatic assumption, for equation (35) a probability distribution

$$P_t(q) \propto \exp -[S_t(q)] \tag{36}$$

with $S_t(q) = q^4 a(t)/4\Gamma$, exists at any time. Therefore a natural ansatz is to assume that (6) becomes

$$R(t_0, \tau) = \left\langle q(t_0, \tau) \frac{\partial S_{t_0}(q)}{\partial q(t_0)} \right\rangle_{t_0}. \tag{37}$$

In figure 3 we show $R(t_0, \tau)$ and $\langle q(t_0, \tau)[\partial S_{t_0}(q)/\partial q(t_0)]\rangle_{t_0}$ against τ, for different t_0, with $a = 1$, $b = -0.5$, $T = 10$ and $\Gamma = 0.5$. The two curves refer to $t_0 = 0$ and $t_0 = T/2$. We see that equation (37) is well obeyed and that the characteristic decay times are different.

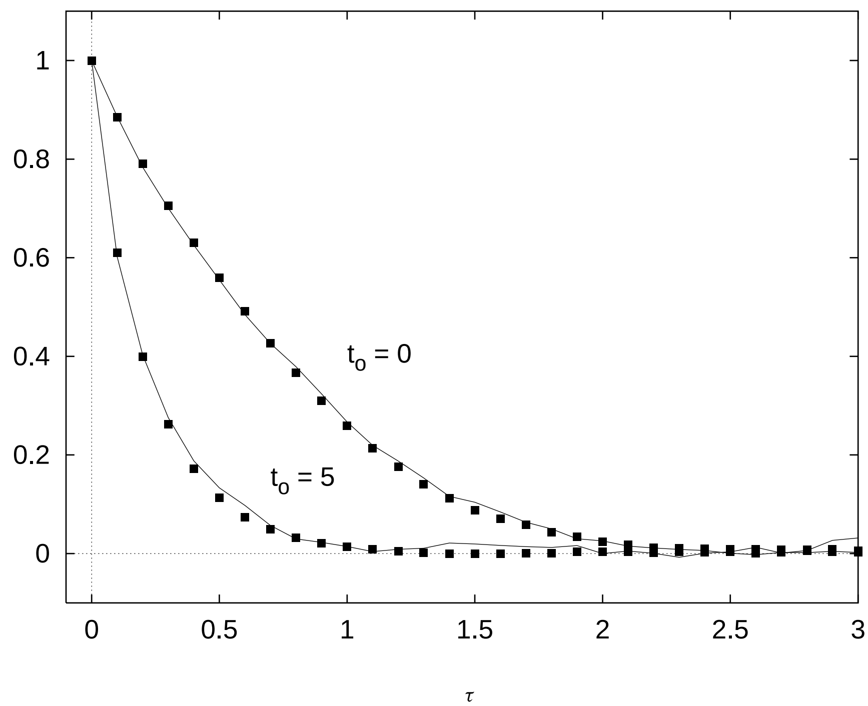

Figure 3. Model (35) and (30) with $a = 1$, $b = -0.9$, $T = 10$ and $\Gamma = 0.5$. The different symbols refer to $R(t_0, \tau)$ (full curve) and $\langle q(t_0, \tau)[\partial S_{t_0}(q)/\partial q(t_0)]\rangle_{t_0}$ (full squares) as functions of τ, for $t_0 = 0$ and $t_0 = T/2 = 5$.

4. Numerical simulations

In this section we report the results obtained for the Lorenz model (8) with the Rayleigh parameter r varying periodically in time according to equation (8). We note that at the value $r = r_c = 166.07$ the standard Lorenz model has a transition from a regular evolution (stable orbit) to a regime of intermittent chaos, and the maximum Lyapunov exponent depends on $r > r_c$ through a scaling law $\lambda \sim \sqrt{r - r_c}$, for $r - r_c \ll 1$ [13]. Since we are interested in chaotic systems with a non-negligible variation of the degree of chaoticity, we chose r_0 close to r_c. Let us recall that λ changes very slowly for r near $r_{cr} = 24.74$, so that a

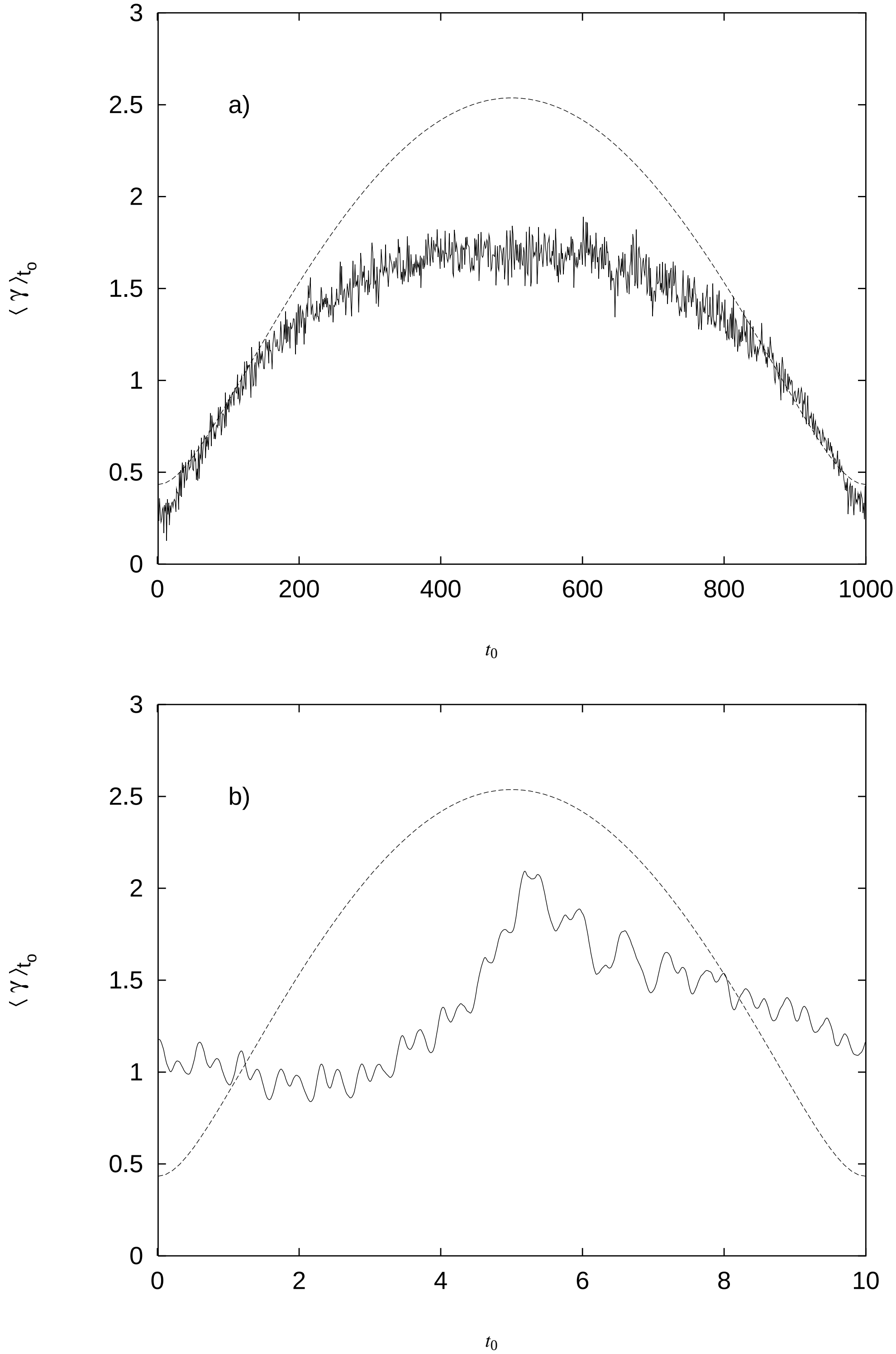

Figure 4. $\langle\gamma\rangle_{t_0}$ as a function of t_0 for the model (7) and (8) with $r_0 = 166.6$, $A = 0, 5$ and (a) $T = 1000$ and (b) $T = 10$. The dashed curve is $\lambda[r(t_0)]$.

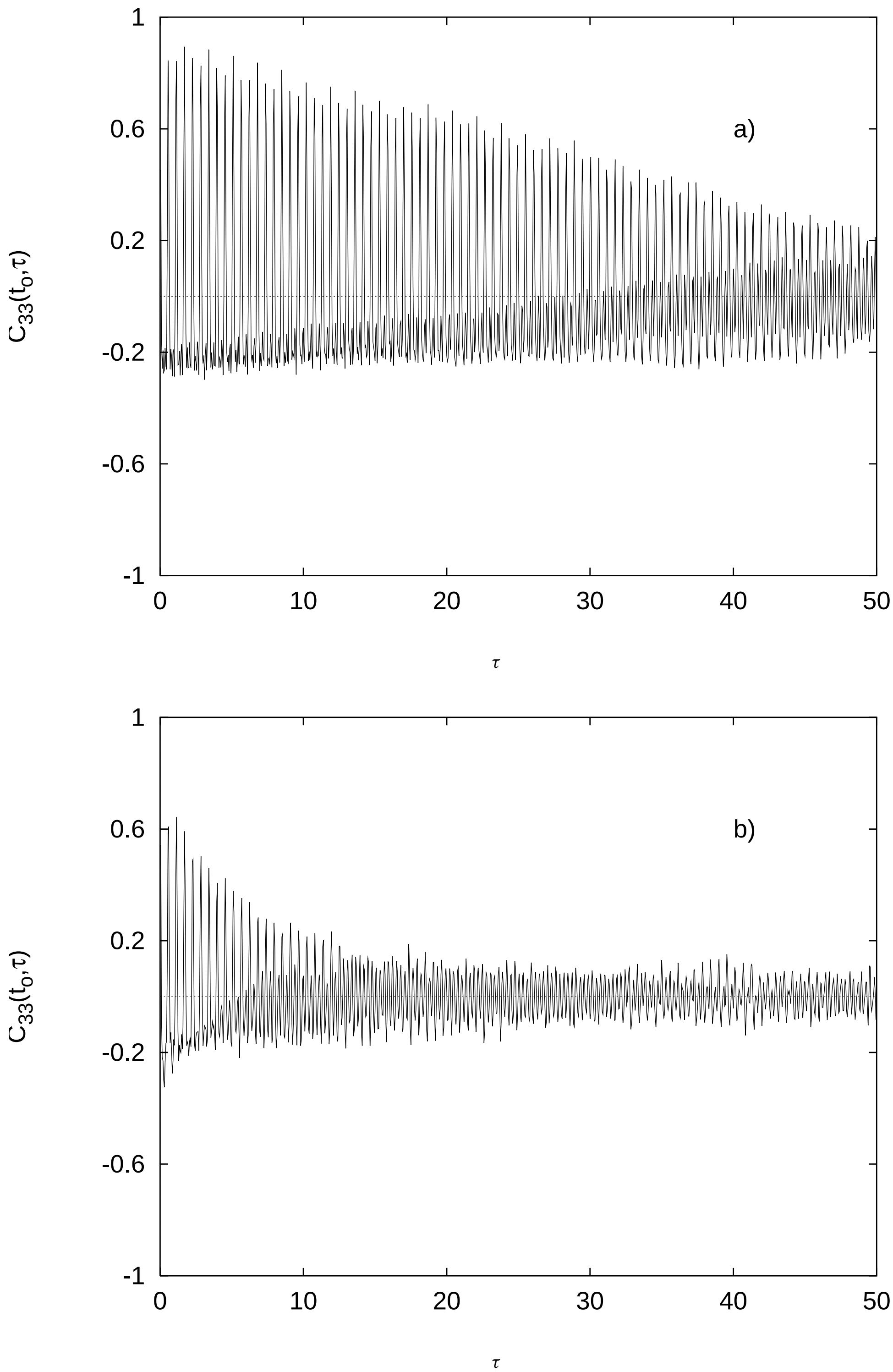

Figure 5. $C_{33}(t_0, \tau)$ as a function of τ for the model (7) and (8) with $r_0 = 166.6$, $A = 0, 5$, $T = 1000$ and (a) $t_0 = 0$ and (b) $t_0 = T/2 = 500$. The dashed curve is $\lambda[r(t_0)]$.

periodic variation of r in this region is not very interesting, at least if $r(t) > r_{cr}$ for any t. We take $r_0 = 166.6$ and $A = 0.5$ and we consider different periods. If T is very large with respect to the period ($\sim$O(1)) of the unstable periodic orbit, the variations of r can be taken as quasi-adiabatic. Therefore, we expect that for large T and for $\tau \ll T$ the local effective Lyapunov exponent of the system varies periodically in time with the same period of r, i.e.

$$\langle\gamma\rangle_{t_0} \approx \lambda[r(t_0)] \sim \sqrt{r(t_0) - r_c}. \tag{38}$$

From figure 4(a) one sees that this approximation holds true if $r(t_0)$ is not too far from r_c. The value of the maximum Lyapunov exponent should be close to the maximum Lyapunov exponent of the stationary Lorenz model computed at r_0, i.e. the average value of r on the whole period.

When T is not long enough, so that one cannot speak anymore of adiabatic attractors, the behaviour of $\langle\gamma\rangle_{t_0}$ has no relation with that of $r(t)$, as shown in figure 4(b).

We discuss now the behaviour of the 'local' correlation functions, defined by equation (17), and of the 'local' response functions, equation (20). From figure 5 one sees rather clearly how the 'local' correlation functions depend rather strongly on the initial time t_0. In particular, the dependence on t_0 of the mean decay time is well evident.

The correlation functions show a rapid oscillating behaviour at times $\sim$O(1), with a typical period of the order of the mean circulation time near the 'ghost' of the stable orbit existing at $r = r_c$, while they decay for long delays, with characteristic times that sensibly depend on the degree of chaos relative to the initial instant.

Let us stress that, because of the highly non-Gaussian nature of the system, there is not a simple direct proportionality between the response R_{ij} and the standard correlation C_{ij}, both in the global and in the local version. This also happens in autonomous systems [3] and is not a pathological behaviour. From very general arguments [3] one can show that the mean response and correlation functions have the same qualitative behaviour, for example the same decaying properties on a large time delay. However, the agreement between R_{ij} and C_{ij} is very poor for moderate delay. This is so because in chaotic systems the error bars, in the numerical computation of the mean response function, increase exponentially with the delay and, when τ is not very large, the mean response has to be compared with a suitable correlation function, see equation (6), which depends on the unknown invariant probability distribution $P(\boldsymbol{x})$.

5. Conclusions

We have discussed how to characterize the behaviour of a chaotic dynamical system when a 'seasonal' effect is present, i.e. when there exist two well separated time scales: the internal one and that of the periodic variation of a control parameter. A proper characterization has been obtained by the introduction of a restricted average for the relevant quantities: correlation functions, response functions and Lyapunov exponents. These selective averages take into account, in an explicit way, the phase of the external period at which the system is observed. We stress that, in the presence of a 'seasonal' effect, the usual global averaged quantities can give only rough information. It is clear that to have sensible local-in-time quantities the time delay τ has to be much smaller than the external forcing period T. We note that in this limit the local-in-time quantities are roughly independent on τ.

Acknowledgments

MF and AV acknowledge the financial support of the Istituto Nazionale di Fisica Nucleare, through the *Iniziativa specifica FI11*. AC, AV and MF acknowledge the financial support of the CNR progetto *Predicibilità in turbolenza geofisica*.

References

[1] Leith C A 1973 *J. Appl. Meteor.* **12** 1066
Leith C A 1978 *Nature* **276** 352
[2] Kubo R, Toda M and Hashitsume N 1985 *Statistical Physics* (Berlin: Springer)
[3] Carnevale G F, Falcioni M, Isola S, Purini R and Vulpiani A 1991 *J. Fluids* A **3** 2247
Falcioni M, Isola S and Vulpiani A 1990 *Phys. Lett.* **144A** 341
[4] Kraichnan R H 1959 *Phys. Rev.* **119** 1181
Leith C A 1975 *J. Atmos. Sci.* **32** 2022
Bell T L 1980 *J. Atmos. Sci.* **37** 1700
[5] Benettin G, Galgani L and Strelcyn J M 1976 *Phys. Rev.* A **14** 2338
[6] Paladin G and Vulpiani A 1987 *Phys. Rep.* **156** 147
[7] Crisanti A, Jensen M H, Paladin G and Vulpiani A 1993 *Phys. Rev. Lett.* **70** 166
Crisanti A, Jensen M H, Paladin G and Vulpiani A 1993 *J. Phys. A: Math. Gen.* **26** 6943
[8] Pikosky A 1993 *Chaos* **3** 225
Paladin G and Vulpiani A 1994 *J. Phys. A: Math. Gen.* **27** 4911
[9] Aurell E, Boffetta G, Crisanti A, Paladin G and Vulpiani A 1996 *Phys. Rev.* E **53** 2337
Boffetta G, Paladin G and Vulpiani A 1996 *J. Phys. A: Math. Gen.* **29** 2291
[10] Crisanti A, Falcioni M, Paladin G and Vulpiani A 1994 *J. Phys. A: Math. Gen.* **27** L597
[11] Lorenz E N 1963 *J. Atmos. Sci.* **20** 130
[12] Wax N (ed) 1954 *Noise and Stochastic Processes* (New York: Dover)
[13] Manneville P and Pomeau Y 1980 *Physica* **1D** 219